Lecture I. (5月14日)

I. ※※

Introduction: The simplest ordinary differential equations (O.D.E.), have the form $\frac{d^n x}{dt^n} = G(t)$, where the derivative of $x = x(t)$ can be of any order, and the right hand side may depend only on the variable t.

Example 1: Consider a mass falling under the influence of constant gravity, such as approximately found on the Earth's surface. Recall Newton's law: $F = ma \Rightarrow m\frac{d^2x}{dt^2} = -mg$, where $x(t)$ is the height of the object above the ground, m is the mass of the object and $g \simeq 9.8\,m/s^2$ is the constant gravitational equation. "$\frac{d^2x}{dt^2} = -g$"
(gravitational acceleration)
is an O.D.E.

Def 1: An ordinary differential equation is an equation of the form: $F(t, y(t), y^{(1)}(t), \cdots, y^{(n)}(t)) = 0$, where $F: I_0 \times I_1 \times \cdots \times I_{n+1} \to \mathbb{R}$ is a smooth function, $y: I_0 \to \mathbb{R}$ is a smooth function, $y^{(j)}(t) = \frac{d^j y}{dt^j}$, $j = 1, 2, \cdots, n$, $I_0, I_1, \cdots, I_{n+1}$ are open intervals of $\mathbb{R}$. n is called the order of the equation.

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Example 2: We rewrite the O.D.E $x^{(2)}(t) = -g$ by using Def 1. Let F: $\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $(t, x_1, x_2, x_3)\to -g-x_3$. $F(t, x_1, x_2, x_3) = -g-x_3$.

$F(t, x(t), x^{(1)}(t), x^{(2)}(t)) = -g-x^{(2)}(t)$.

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Def 2: An O.D.E. $F(t, y(t), y^{(1)}(t), \dots, y^{(n)}(t))$ is called linear if $F(t, y(t), \dots, y^{(n)}(t))$ $= a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y^{(1)}(t) + a_0(t)y(t) + r(t)$, where $a_j(t)\in C^\infty(I_0)$, $j=0,1,\dots,n$, $r(t)\in C^\infty(I_0)$. A linear O.D.E. is called homogeneous if $r(t)\equiv 0$. A linear O.D.E. is said to have constant coefficient if $a_j(t)$ is a constant function, for every $j=0,1,2,\dots$.

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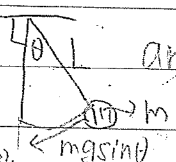
Example 3: Oscillating pendulum (單擺運動):  arc length: $L\theta(t)$, $\underbrace{a}_{\text{Acceleration}} = L\theta^{(2)}(t)$

By Newton's law, $-mg\sin\theta(t) = mL\theta^{(2)}(t)$. $\Rightarrow \theta^{(2)}(t) + \frac{g}{L}\sin\theta(t) = 0$.

This is a non-linear O.D.E (not simple harmonic motion) (簡諧運動). When $\theta(t)$ is small, $\sin\theta(t)\sim\theta(t)$; the solution of the linear O.D.E. $\theta^{(2)}(t) + \frac{g}{L}\theta(t) = 0$ is an approximate solution of the non-linear O.D.E. $\theta^{(2)}(t) + \frac{g}{L}\sin\theta(t) = 0$.

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O.D.E. plays an important role in science, applied math, physics, engineering, biology, economics. We will learn

Lecture I.

1. (1st order O.D.E.), $y' = f(t,y)$. Chapter 2. (5/14, 5/16, 5/17) 5)
(2.1, 2.2, 2.6)

2. (2nd order linear O.D.E.) Chapter 3, $a(t)y''(t)+b(t)y'(t)+c(t)y(t) = f(t)$. (5/20, 5/21, 5/23, 5/24) 6)
(3.1, 3.2, 3.3, 3.4, 3.5)

3. (higher order linear O.D.E.) Chapter 4, $a_n(t)y^{(n)}(t)+\cdots+a_0(t)y(t) = f(t)$. (5/27, 5/28, 5/30) 11)
(4.1, 4.2, 4.3)

4. System of 1st order O.D.E., Chapter 7, $\begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = A(t)\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$, where $A(t)$ is a t-dependent $n \times n$ matrix. (5/30, 5/31, 6/3, 6/4, 6/6)
(7.1, 7.2, 7.3, 7.4, 7.5)

5. The Laplace transform, Chapter 6. (6/17, 6/18, 6/20, 6/21) 6

6. Prove existence and uniqueness. (6/24, 6/25, 6/27) 4

During the week 10 June ~ 14 June, I have to participate a conference. We do not have lectures at this week but we will have mid-term examination at 14 June. 10 June, 11 June, 13 June (Total 180 mins).

During 5/14 ~ 6/21, each lecture will has 50 minutes.
(27 lecture) No break 4:30 ~ 6:10

6 June No tutorial, 13 June (two tutorial) $27 \times 5 + 45 = 180$.
Lecture

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IV. ✱✱

§2: First order Differential equations: In general, a first order O.D.E. can be written as $\frac{dy}{dt} = f(t,y)$ for some function f. Solve this O.D.E. means that find a t-dependent function $y(t)$ such that $y'(t) = f(t, y(t))$. In this Chapter, we are going to solve the O.D.E. above explicitly with $\begin{cases} \text{I. } f(t,y) = p(t)y + q(t), \\ \text{II. } f(t,y) = g(y)h(t). \end{cases}$ I is linear, II can be non-linear.

§2.1: Let $p(t), q(t) \in C^\infty((a, +\infty))$. Consider the O.D.E.: $y'(t) +$
(Method of integrating factor)
$p(t)y(t) = q(t)$ with initial condition $y(a) = b$. How to solve this equation? Let $P(t)$ be an anti-derivative of $p(t)$, i.e. $P'(t) = p(t)$.

$$\Rightarrow e^{P(t)}(y'(t) + p(t)y(t)) = e^{P(t)}q(t) = \frac{\partial}{\partial t}\left(e^{P(t)}y(t)\right) = e^{P(t)}(P'(t)y(t) + y'(t)).$$

$$\Rightarrow e^{P(t)}y(t) = \int_a^t \frac{\partial}{\partial s}\left(e^{P(s)}y(s)\right)ds + e^{P(a)}y(a) = \int_a^t e^{P(s)}q(s)\,ds + e^{P(a)}y(a).$$

We get $y(t) = e^{-P(t)}\int_a^t e^{P(s)}q(s)\,ds + y(a)$. If we do not know the initial data, then $y(t) = e^{-P(t)}\int_a^t e^{P(s)}q(s)\,ds + C_1$, C_1 can be any constant. The solution is not unique.

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V. **

Example 2.1: Solve $y'(t)+\frac{1}{2}y(t)=\frac{1}{2}e^{\frac{t}{3}}$. $\frac{1}{2}t$ is the anti-derivative of $\frac{1}{2}$. $\Rightarrow$ $e^{\frac{1}{2}t}(y'(t)+\frac{1}{2}y(t))=\frac{1}{2}e^{\frac{5t}{6}}=\frac{\partial}{\partial t}(e^{\frac{1}{2}t}y(t))$. $\Rightarrow$ $e^{\frac{1}{2}t}y(t)=\int_0^t \frac{1}{2}e^{\frac{5s}{6}}ds+C_1=\frac{3}{5}e^{\frac{5}{6}t}+C_1$ $\Rightarrow$ $y(t)=\frac{3}{5}e^{\frac{1}{3}t}+C_1e^{-\frac{t}{2}}$, C, C_1 are constants. ///

Example 2.2: Solve $ty'(t)+2y(t)=4t^2$ on $\{t\geq 1\}$ with $y(1)=2$.

$\because t\geq 1$. $\Rightarrow$ $y'(t)+\frac{2}{t}y(t)=4t$. $\frac{\partial}{\partial t}\log t^2=\frac{2}{t}$. $\Rightarrow$ $e^{\log t^2}(y'(t)+\frac{2}{t}y(t))$ $=e^{\log t^2}4t=4t^3=\frac{\partial}{\partial t}(t^2y(t))$. $\Rightarrow$ $t^2y(t)=\int_1^t 4s^3ds+2=t^4+1$. $\Rightarrow$ $y(t)=t^2+\frac{1}{t^2}$ ///

§2.2: Separable Equations. Suppose that we are given an O.D.E. $y'(t)=g(y(t))h(t)$ on $\{t\geq 0\}$ with initial condition $y(0)=b$, where $1/g$ is locally integrable on $[0,+\infty)$.

Def 2.3 For a function $h: [0,+\infty)\to\mathbb{R}$, we say that h is locally integrable on $[0,+\infty)$ if for all $s\in[0,+\infty)$, we have $\int_0^s |h(s)|ds<+\infty$. ///

Example 2.4: Let $h(t)=\frac{1}{t}$, then $\int_0^1\frac{1}{t}=+\infty$, $\frac{1}{t}$ is not locally integrable. Let $h(t)=\frac{1}{\sqrt{t}}$, then $\int_0^s\frac{1}{\sqrt{t}}dt<+\infty$, for all $s\in[0,+\infty)$. $\frac{1}{\sqrt{t}}$ is locally integrable. ///

VI. ⁂

Lecture I.

Let G be an anti-derivative of $\frac{1}{g}$. $G(t)=\int_0^t \frac{1}{g(s)}ds. \Rightarrow G'(t)=\frac{1}{g(t)}$.

$\Rightarrow \frac{1}{g(y(t))}y'(t)=h(t) \Rightarrow G'(y(t))y'(t)=h(t) \Rightarrow \frac{\partial}{\partial t}(G(y(t)))=h(t) \Rightarrow G(y(t))$

$=\int_0^t h(s)ds+G(y(0))=\int_0^t h(s)ds+G(b). \Rightarrow y(t)$ can be solved if

the inverse function of G is known.

Example 2.5: Let $y(x)$ be a solution of the O.D.E $y'(x)=\frac{x^2}{1-y^2(x)}$.

Show that $x, y(x)$ satisfies $x^3+y^3(x)-3y(x)=C$ for some constant C.

pf: $y'(x)=g(y(x))h(x)$, $h(x)=x^2$, $g=\frac{1}{1-x^2}$, $\frac{1}{g}=1-x^2 \Rightarrow G(x)=x-\frac{1}{3}x^3$ is

an anti-derivative of $\frac{1}{g} \Rightarrow (G(y(x)))'=h(x) \Rightarrow (y(x)-\frac{1}{3}y^3(x))'=x^2$

$\Rightarrow y(x)-\frac{1}{3}y^3(x)=\int_0^x x^2dx+C_1=\frac{1}{3}x^3+C_1 \Rightarrow x^3+y^3(x)-3y(x)=C.$

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Example 2.6: Solve the O.D.E $y'(x)=\frac{3x^2+4x+2}{2(y(x)-1)}$ with $y(0)=-1$. $y'(x)=g(y(x))h(x)$

where $g(x)=\frac{1}{2x-2} \Rightarrow \frac{1}{g}=2x-2$ and $G(x)=x^2-2x$ is an anti-derivative of $\frac{1}{g}$

$\Rightarrow (G(y(x)))'=3x^2+4x+2 \Rightarrow G(y(x))=\int_0^x (3x^2+4x+2)dx+C$

$\Rightarrow G(y(x))=x^3+2x^2+2x+C=y^2(x)-2y(x) \Rightarrow C=3$

$\Rightarrow y^2(x)-2y(x)=x^3+2x^2+2x+3 \Rightarrow (y(x)-1)^2=x^3+2x^2+2x+4$

$y(x)=1-\sqrt{x^3+2x^2+2x+4}$

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VII. ※※

Def 2.7 (Integral curves): Let $F=(F_1,\cdots,F_n)$ be a vector field. A parametric curve $X(t)=(X_1(t),\cdots,X_n(t))$ is said to be an integral curve of F if $X_1'(t)=F_1(X_1(t),\cdots,X_n(t)),\cdots,X_n'(t)=F_1(X_1(t),\cdots,X_n(t))$.

In geometry and physics, it is important to find an integral curve of a given vector field F. Len $n=2$. To find integral curve is reduced to solve the equation $\begin{cases} x'(t)=F(x(t),y(t)) \\ y'(t)=G(x(t),y(t)) \end{cases}$. At $t_0\in\mathbb{R}$, assume that $x'(t_0)\neq 0$. Recall (advanced calculus), near t_0, $x: t\to x(t)$ is invertible. Let g be the inverse of $x(t)$, that is, $g(x(t))=t$. Then,

$y(t)=y(g(x(t))) \Rightarrow \underset{y'(x)}{y'(t)}=\frac{dy}{dx}\frac{dx}{dt}=\frac{dy}{dx}F(x,y)=G(x,y) \Rightarrow$ The equation is reduced to $\frac{dy}{dx}=\frac{G(x,y)}{F(x,y)}$. Formally, $\frac{dx}{dt}=F(x,y)$, $\frac{dy}{dt}=G(x,y)$,

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{G(x,y)}{F(x,y)}$.

Example 2.8: Find the integral curve of the vector field $F(x,y)=(4+y^3, 4x-x^3)$ passing through $(0,1)$. Find $(x(t),y(t))$ so that $x'(t)=4+y^3(t)$, $y'(t)=4x(t)-x^3(t)$. Assume $(x(t_0),y(t_0))=(0,1)$.

Lecture II.

III. ※

Notation: Let C be a piecewise smooth curve starting at q_0 and ending at q_1.

If $q_0 = q_1$, then C is called a closed curve. We can always find a piecewise smooth function $(x(t), y(t)) : [0, s] \to D$ such that $(x(0), y(0)) = q_0$, $(x(s), y(s)) = q_1$, $x'(t)^2 + y'(t)^2 = 1$, for every $t_0 \in [0, s]$, $(x(t), y(t))$ smooth at t_0, $C = \{(x(t), y(t)) \mid t \in [0, s]\}$. Then $\oint_C F \cdot dr =$

$$\int_0^s (M(x(t), y(t))\, x'(t) + N(x(t), y(t))\, y'(t))\, dt.$$

$\oint_C$: path integral.

We write $C = (x(t), y(t)) : [0, s] \to D$.

pf: $(1) \Rightarrow (2)$ Let $C = (x(t), y(t)) : [0, s] \to D$ be a smooth closed curve.

Note that $(x(0), y(0)) = (x(s), y(s))$. $F = (M, N) = (\varphi_x, \varphi_y)$, $\varphi \in C^\infty(D)$.

Then, $\oint_C F \cdot dr = \int_0^s \left[\frac{\partial \varphi}{\partial x}(x(t), y(t))\, x'(t) + \frac{\partial \varphi}{\partial y}(x(t), y(t))\, y'(t)\right] dt = \int_0^s \frac{\partial}{\partial t}\left[\varphi(x(t), y(t))\right] = \varphi(x(s), y(s)) - \varphi(x(0), y(0)) = 0$.

$(2) \Rightarrow (3)$ Let $C_0 = (x_0(t), y_0(t)) : [0, s_0] \to D$, $C_1 = (x_1(t), y_1(t)) : [0, s_1] \to D$ be two smooth curves such that $(x_0(0), y_0(0)) = (x_1(0), y_1(0)) = p_0$, $(x_0(s_0), y_1(s_0)) = (x_1(s_1), y_1(s_1)) = p_1$. Let

$C : [0, s_0 + s_1] \to D$ be a piecewise smooth curve in D given by $t \to (x(t), y(t))$

$t \in [0, s_0] \to (x_0(t), y_0(t))$, $t \in [s_0, s_0 + s_1] \to (x_1(-t + s_1 + s_0), y_1(-t + s_1 + s_0))$.

Then $(x(0), y(0)) = p_0$, $(x(s_0 + s_1), y(s_0 + s_1)) = p_0$. C is a piecewise closed curve.

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Lecture II.

Note that $(x_0(s_0), y_0(s_0)) = (x_1(-s_0+s_1+s_0), y_1(-s_0+s_1+s_0)) = (x_1(s_1), y_1(s_1))$ $= p_1$. Then, $\oint_C F.dr = \int_0^{s_0+s_1} (M(x(t), y(t))\, x'(t) + N(x(t), y(t))\, y'(t))\, dt$

$= \int_0^{s_0} (M(x_0(t), y_0(t))\, x_0'(t) + N(x_0(t), y_0(t))\, y_0'(t))\, dt + \int_{s_0}^{s_0+s_1} (M(x_1(-t+s_0+s_1),$ $y_1(-t+s_0+s_1))\, -x_1'(-t+s_0+s_1) + N(x_1(-t+s_0+s_1), y_1(-t+s_0+s_1))\, -y_1'(-t+s_0+s_1))\, dt$

$= \oint_{C_0} F.dr - \oint_{C_1} F.dr = 0 \Rightarrow \oint_{C_0} F.dr = \oint_{C_1} F.dr$. (3) $\Rightarrow$ (1) Fix $(a,b) \in$

Let $\varphi : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function on D given by $\varphi(x,y) := \oint_{C[(a,b),(x,y)]} F.dr$, where $C[(a,b),(x,y)]$ denotes a piecewise smooth curve starting at (a,b), ending at (x,y). By (3), φ is well-defined. Fix $(x_0,y_0) \in D$. Let $h > 0$ be a small num

$\varphi(x_0+h, y_0) - \varphi(x_0, y_0) = \int_{C[(x_0,y_0),(x_0+h,y_0)]} F.dr$. Take $C[(x_0,y_0),(x_0+h,y_0)] : t \in [0,h] \mapsto (x_0+t, y_0) \in$

Then, $\varphi(x_0+h, y_0) - \varphi(x_0, y_0) = \int_0^h M(x_0+t, y_0)\, dt \Rightarrow \varphi_x(x_0,y_0) = \lim_{h \to 0} \frac{\varphi(x_0+h, y_0) - \varphi(x_0,}{h}$

$= \frac{1}{h}\int_0^h M(x_0+t, y_0)\, dt = M(x_0, y_0)$. Similarly, $\varphi_y = N$.

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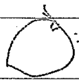
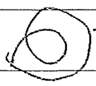
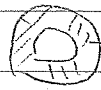
<u>Def 2.16</u>: Let D be a path connected domain in $\mathbb{R}^2$. We say that D is simply connected if every simple closed curve can be smoothly shrunk to a point in D without any part ever passing out of D. ▨

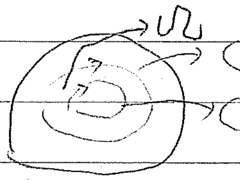
Let $C = (x(t), y(t)) : [0, s] \to D$ be a closed curve. C is called simple if $(x(t_0), y(t_0)) \neq (x(t_1), y(t_1))$, for every $t_0, t_1 \in (0, s)$ with $t_0 \neq t_1$.

 → simple  → not simple  → simply connected.  → not simply connect

Roughly speaking, a simply connected domain is one without holes on it.

Thm 2.17: Let $D \subset \mathbb{R}^2$ be a simply connected domain. Let $M, N \in C^\infty(D)$. If $M_y = N_x$ on D, then $F = (M, N)$ is conservative. ▨

<u>pf</u>: By <u>Thm 2.15</u>, we only need to show that $\oint_C F.dr = 0$, for every piecewise smooth curve C.



$\oint_C F.dr = \int_C M dx + N dy$

By stoke's thm, (Gauss-Green) $\oint_C F.dr - \oint_{C_\delta} F.dr = \int_\Omega (M_y - N_x) dx dy = 0$. $C_\delta \to \{P\}$.

Hence, $\oint_C F.dr = 0$. The theorem follows. ▨

Key point: By stoke's thm, (Gauss-Green) $\oint_C F.dr = \oint_{C_\delta} F.dr$ if $M_y = N_x$ on D. ▨

$\left(\int_V \nabla . F \, dV = \int_{\partial V} F . dS\right)$

Lecture III.

II. ※※

Example 2.18: Let $D = \mathbb{R}^2 / \{(0,0)\}$. D is not simply connected. Let $M(x,y) = \frac{-y}{x^2+y^2}$, $N(x,y) = \frac{x}{x^2+y^2}$. Then $M_y = N_x = \frac{y^2-x^2}{(x^2+y^2)^2}$ on D. Hower we can not find $\varphi \in C^\infty(D)$ such that $\varphi_x = \frac{-y}{x^2+y^2}$, $\varphi_y = \frac{x}{x^2+y^2}$. Let $C_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$. $C_1 = (\cos t, \sin t) : [0, 2\pi] \to D$. C_1 is a closed curve in D.

$\oint_{C_1} F.dr = \int_0^{2\pi} (M(\cos t, \sin t)(-\sin t) + N(\cos t, \sin t)\cos t)dt = \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt$

$= 2\pi$. But if we can find such φ, then F is conservative and $\oint_C F.dr = 0$. Hence, we can not find $\varphi \in C^\infty(D)$ such that $\varphi_x = M$, $\varphi_y = N$. ▨

Def 2.19: An ODE of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is called exact, where M, N are smooth function, if there exists a smooth function φ, called the potential function, such that $\varphi_x = M$, $\varphi_y = N$. ▨

Consider the O.D.E. $M(x,y) + N(x,y)\frac{dy}{dx} = 0$. It is important to know that how to solve this O.D.E. Let $D \subset \mathbb{R}^2$ be a simply connected domain and assume that $M, N \in C^\infty(D)$. We want to solve the O.D.E on D.

Lecture III.

Case I. If $M_y = N_x$ on D. By Thm 2.17, there is a $\varphi \in C^\infty(D)$ such that $\varphi_x = M$, $\varphi_y = M$. Then the O.D.E becomes $\varphi_x(x,y) + \varphi_y(x,y)\frac{dy}{dx} = 0$. Fix $(x_0, y_0) \in D$ and assume that $\varphi_y(x_0, y_0) \neq 0$. Let $\varphi(x_0, y_0) = C_1$. By implici function theorem, near x_0, we can find a smooth function $y(x)$ such that $\varphi(x, y(x)) = C_1 \Rightarrow \varphi_x(x, y(x)) + \varphi_y(x, y(x))\, y'(x) = 0$. $y(x)$ is the solution of the O.D.E above. ////

Case II. If $M_y \neq N_x$. Fact (Theorem): There is a $\mu \in C^\infty(D)$, $\mu \neq 0$, such that $(\mu M)_y = (\mu N)_x$. Such a μ is called an integrating factor. We can prove that there is a such μ but it is hard to find explicit expression. $\Rightarrow \mu M(x,y) + \mu N(x,y)\frac{dy}{dx} = 0$. We then reduce to Case I. ////

Example 2.20: Solve $(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)\frac{dy}{dx} = 0$.

Let $M(x,y) = y\cos x + 2xe^y$, $N(x,y) = \sin x + x^2 e^y - 1$. Then $M_y = \cos x + 2xe^y = N_x(x,y)$. The ODE is exact. How to find φ such that $\varphi_x = M$, $\varphi_y = N$.

$\varphi_x(x,y) = M \Rightarrow \varphi(x,y) = \Phi(y) + \int_0^x \varphi_x(x,y)\,dx = \Phi(y) + \int_0^x (y\cos x + 2xe^y)\,dx =$

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Lecture III.

$= \Phi(y) + y\sin x + x^2 e^y$, for some smooth function Φ. $\Rightarrow \varphi_y = \Phi'(y)$ $+ \sin x + x^2 e^y = N = \sin x + x^2 e^y - 1 \Rightarrow \Phi'(y) = -1 \Rightarrow \Phi(y) = -y + C$, C constant $\Rightarrow \varphi(x,y) = y\sin x + x^2 e^y - y + C$. $\varphi_y(0,1) = -1 \neq 0$. By implicit function theorem, near $x=0$, there is a smooth function $y(x)$ such that $\varphi(x, y(x)) = \varphi(0,1) \Rightarrow y(x)$ solves the O.D.E. above. $\boxtimes$

Example 2.21: Solve $(3xy+y^2) + (x^2+xy)\frac{dy}{dx} = 0$. Let $M(x,y) = 3xy+y^2$, $N(x,y) = x^2+xy$. Then, $M_y = 3x+2y$, $N_x = 2x+y$, $M_y \neq N_x$. We look for an integrating factor μ so that $(\mu M)_y = (\mu N)_x$. $\Rightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0 \Rightarrow (3xy+y^2)\mu_y - (x^2+xy)\mu_x + (x+y)\mu = 0$. The integrating factor μ is not unique. Let $\mu = x \Rightarrow (3x^2y+xy^2) + (x^3+x^2y)\frac{dy}{dx} = 0$.

Let $\hat{M} = 3x^2y + xy^2$, $\hat{N} = x^3 + x^2y$. $\hat{M}_y = \hat{N}_x$. Find φ so that $\varphi_x = \hat{M}$, $\varphi_y = \hat{N}$.

$\varphi = \Phi(y) + \int_0^x (3x^2y + xy^2)\,dx \Rightarrow \varphi_x = 3x^2y + xy^2$; $\varphi_y = \Phi'(y) + \int_0^x (3x^2 + 2xy)\,dx$ $= \Phi'(y) + x^3 + x^2y = x^3 + x^2y$, $\Rightarrow \Phi'(y) = 0 \Rightarrow \Phi(y) = C \Rightarrow \varphi = x^3y + \frac{x^2y^2}{2} + C$.

At $(1,1)$, $\varphi_y(1,1) \neq 0$. Near $(1,1)$, there is a smooth function $y(x)$ such that $x^3y(x) + \frac{x^2y^2(x)}{2} = \frac{3}{2} \Rightarrow 3x^2y(x) + x^3y'(x) + xy^2(x) + x^2yy'(x) = 0 \Rightarrow (3x^2y + xy^2) + (x^3 + x^2y)\,y'(x) = 0$.